The asymptotic iteration method for the angular spheroidal eigenvalues with arbitrary complex size parameter c

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Abstract

The asymptotic iteration method is applied, to calculate the angular spheroidal eigenvalues $\lambda_\ell^m(c)$ with arbitrary complex size parameter c. It is shown that, the obtained numerical results of $\lambda_\ell^m(c)$ are all in excellent agreement with the available published data over the full range of parameter values ℓ, m, and c. Some representative values of $\lambda_\ell^m(c)$ for large real c are also given.

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1 Introduction

The solution of the spheroidal wave equation is a very old subject, but it still an important theme in the existing literature. The importance of this equation arises in many areas of physics. For instance, it plays an important role in the study of light scattering in optics [1-3], nuclear modeling [4], signal processing and communication theory [5], electromagnetic modeling [6], and in finding the electromagnetic induction (EMI) response of canonical objects at magnetoquasistatic frequencies [7]. Applications utilizing complex c include for example, light scattering from spheroidal particles, and spheroidal antennas enveloped in a plasma medium.

Attempts to find rapid, and accurate eigenvalues $\lambda_\ell^m(c)$ of the angular spheroidal wave equation for large size parameter c^2 (assumed real) have been ongoing. Flammer summarizes the work (up to 1957 [8]), and documents the asymptotic expansions for $\lambda_\ell^m(c)$ of the angular spheroidal wave equation. Since that time, serious attempts for this case were made by many authors. Slepian [9], and Streifer [10] derived uniform asymptotic expansions for the spheroidal functions and their eigenvalues, which were further developed by des Cloiseaux, and Mehta [11], and Dunster [12]. Other asymptotic results based on WKB methods have been obtained by Sink, and Eu [13]. Recently, asymptotic expansions of $\lambda_\ell^m(c)$ for large l, and c have been proposed by Guimarães [14], de Moraes and Guimarães [15], and the work of Do-Nhat [16, 17] summarizes and provides more details of Flammer's expansions for $\lambda_\ell^m(c)$.

Nevertheless, because of the complexity of the angular spheroidal wave equation with arbitrary complex size parameter $c = c_r + c_i i$, where $c_r = Re\{c\}$, and $c_i = Im\{c\}$, evaluation of $\lambda_\ell^m(c)$ in this regime has been much less studied.

Very recently, Barrowes et al. [18] compute the asymptotic expansions of $\lambda_\ell^m(c)$

with arbitrary complex size parameter c in the asymptotic regime of large $\mid c \mid$ with l, and m fixed. On the other hand, few packages have been developed for the computation of the angular spheroidal eigenvalues $\lambda_\ell^m(c)$ with arbitrary complex size parameter $c = c_r + c_i i$. Thompson [19], Li et al. [20, 21], and Falloon et al. [22] are of the most recent ones.

Those attempts to obtain the eigenvalues $\lambda_\ell^m(c)$ with arbitrary complex size parameter $c = c_r + c_i i$ are rely heavily on power series expansions, and complicated recurrence relations. Accurate results in those works are obtainable at the expense of extensive mathematical, and numerical manipulations, thus obscuring the physical analysis of the corresponding system.

The present work applies the asymptotic iteration method (AIM) [23, 24], for the computation of the angular spheroidal eigenvalues $\lambda_\ell^m(c)$ with arbitrary complex size parameter $c = c_r + c_i i$. This method was applied by Barakat et al. [25] to compute the angular spheroidal eigenvalues $\lambda_\ell^m(c)$ with real c^2, and for the eigenenergies of the anharmonic oscillator potential [26]. The implementation of this method was straightforward, and the results were sufficiently accurate for practical purposes. Most importantly, the numerical computation of the angular spheroidal eigenvalues using this method was quite simple, fast, and the eigenvalues were satisfying a simple ordering relation. Therefore, one can unambiguously select the correct starting eigenvalue.

Furthermore, AIM was quite flexible in the sense that, it is applicable to any parameter value involved like ℓ, m, and c. It also handles $\lambda_\ell^m(c)$ with large ℓ, and c which poses many numerical instabilities to some of the previously mentioned methods. Therefore, the main motivation of the present work is to overcome the shortcomings of those approaches, and to formulate an elegant algebraic approach

to yield a fairly simple analytic formula which will give rapidly the eigenvalues with high accuracy.

In this spirit, this paper is organized as follows. In Sec. 2 the asymptotic iteration method for the angular spheroidal wave equation is outlined. The analytical expressions for asymptotic iteration method are cast in such a way that allows the reader to use them without proceeding into their derivation. In Sec. 3 we present our numerical results compared with other works, and then we conclude and remark therein.

2 Formalism of the asymptotic iteration method for the angular spheroidal wave equation

The angular spheroidal wave equation, with which we shall be concerned, is

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{\ell,m}(c;\eta)\right]+\left[(\lambda_\ell^m(c))^2-c^2\eta^2-\frac{m^2}{(1-\eta^2)}\right]S_{\ell,m}(c;\eta)=0. \qquad (1)$$

The parameter c, which is related to the ellipticity of the spheroidal coordinate surfaces, is allowed to be a complex variable in the present work. Consequently, there is no need to make the usual distinction between the prolate, and oblate forms of the spheroidal wave equation, and the prolate form equation (1) is adopted for definiteness. The other parameters $\lambda_\ell^m(c)$, and m are separation constants.

The second arises as a wave number for the polar angle of spheroidal coordinates and, as usual, is required to be a nonnegative integer; $\ell \geq m$ is an integer enumerating the eigenvalues, and functions.

The spheroidal wave functions $S_{\ell,m}(c;\eta)$ are defined to be the solutions of equation (1) that are finite at the two end points $\eta = \pm 1$ of the range of the independent variable. These finiteness can be satisfied only for certain eigenvalues $\lambda_\ell^m(c)$, which

depend on the values of c once a specific value of m has been chosen.

The simplest case is that of $c = 0$, for which the function $S_{\ell,m}(c;\eta)$ reduces to the associated Legendre function, and $(\lambda_\ell^m(c))^2 = \ell(\ell+1)$ is its eigenvalues.

Here, the integer ℓ labels successive eigenvalues for fixed m. When $\ell = m$ we have the lowest eigenvalue, and the corresponding eigenfunction has no nodes in the interval $-1 \leq \eta \leq 1$. When $\ell = m + 1$ we have the next eigenvalue, and the eigenfunction has one node inside (-1, 1); and so on. A similar situation holds for the general case $c \neq 0$.

In order to apply the AIM for the general case $c \neq 0$, we have to investigate the behavior of the solution near the singular points $\eta = \pm 1$. Substituting a power series expansion of the form

$$S_{\ell,m}(c;\eta) = (1-\eta^2)^\alpha \sum_{k=0}^{\infty} a_k (1-\eta^2)^k, \tag{2}$$

into equation (1), we find that the regular solution has $\alpha = m/2$. Without loss of generality, we can take $m \geq 0$ since $m \to -m$ is a symmetry of the equation. Therefore, we get an equation that is more tractable to the method if we factor out this behavior. Accordingly, we set

$$S_{\ell,m}(c;\eta) = (1-\eta^2)^{m/2} y_{\ell,m}(c;\eta), \tag{3}$$

then the new function $y_{\ell,m}(c;\eta)$ will satisfy a second-order homogenous linear differential equation of the form

$$(1-\eta^2)\frac{d^2 y_{\ell,m}(c;\eta)}{d\eta^2} - 2(m+1)\eta\frac{dy_{\ell,m}(c;\eta)}{d\eta} + (\varepsilon - c^2\eta^2)y_{\ell,m}(c;\eta) = 0, \tag{4}$$

where

$$\varepsilon \equiv (\lambda_\ell^m(c))^2 - m(m+1). \tag{5}$$

Both equations (1), and (4) are invariant under the replacement $\eta \to -\eta$. Thus the functions $S_{\ell,m}(c;\eta)$, and $y_{\ell,m}(c;\eta)$ must also be invariant, except possibly for an overall scale factor.

The systematic procedure of the AIM begins by rewriting equation (4) in the following form

$$y^{''}_{\ell,m}(c;\eta) = \lambda_0(\eta) y^{'}_{\ell,m}(c;\eta) + s_0(\eta) y_{\ell,m}(c;\eta), \tag{6}$$

where

$$\lambda_0(\eta) = \frac{2(m+1)\eta}{(1-\eta^2)}, \; and \quad s_0(\eta) = -\frac{\varepsilon - c^2\eta^2}{(1-\eta^2)}, \tag{7}$$

and following the technique of AIM [23, 25], that will lead to a general solution of equation (6):

$$y_{\ell,m}(c;\eta) = exp\left(-\int^{\eta} \beta(\eta^{'})d\eta^{'}\right)\left[C_2 + C_1 \int^{\eta} exp\left(\int^{\eta^{'}} \{\lambda_0(\eta^{''}) + 2\beta(\eta^{''})\}d\eta^{''}\right) d\eta^{'}\right]. \tag{8}$$

If for some $n > 0$,

$$\beta(\eta) \equiv \frac{s_n(\eta)}{\lambda_n(\eta)} = \frac{s_{n-1}(\eta)}{\lambda_{n-1}(\eta)}, \tag{9}$$

with

$$\lambda_n(\eta) = \lambda^{'}_{n-1}(\eta) + s_{n-1}(\eta) + \lambda_0(\eta)\lambda_{n-1}(\eta), \; and \quad s_n(\eta) = s^{'}_{n-1}(\eta) + s_0(\eta)\lambda_{n-1}(\eta). \tag{10}$$

For sufficiently large n, we can now introduce the termination condition of the method, which in turn, yields the angular spheroidal eigenvalues $\lambda^m_\ell(c)$

$$\delta_n(\eta) \equiv s_n(\eta)\lambda_{n-1}(\eta) - s_{n-1}(\eta)\lambda_n(\eta) = 0. \tag{11}$$

3 Numerical results for the angular spheroidal eigenvalues $\lambda_\ell^m(c)$

Within the framework of the AIM mentioned in the above section, the angular spheroidal eigenvalues $\lambda_\ell^m(c)$ are calculated by means of equation (11). To obtain the eigenvalues $\lambda_\ell^m(c)$, the iterations should be terminated by imposing a condition $\delta_n(\eta)= 0$ as an approximation to equation (11). On the other hand, for each iteration, the expression $\delta_n(\eta) = s_n(\eta)\lambda_{n-1}(\eta) - s_{n-1}(\eta)\lambda_n(\eta)$ depends on two variables: $\lambda_\ell^m(c)$, and η. The calculated eigenvalues $\lambda_\ell^m(c)$ by means of this condition should, however, be independent of the choice of η. Nevertheless, the choice of η is observed to be critical only to the speed of the convergence to the eigenvalues, as well as for the stability of the process. In this work it is observed that, the best starting value for η is the value at which the effective potential of equation (1) takes its minimum value. For this purpose, it is necessary to perform the variable change $\eta \rightarrow tanh(x)$, mapping the finite interval (-1,1) into the infinite one $(-\infty, \infty)$, and then equation (1) can be rewritten as

$$-\frac{d^2}{dx^2}S_{\ell,m} + V_{eff}(x)S_{\ell,m} = -m^2 S_{\ell,m}, \tag{12}$$

where the effective potential $V_{eff}(x)$ is

$$V_{eff}(x) = -\left[(\lambda_\ell^m(c))^2 - c^2\right] sech^2(x) - c^2 sech^4(x). \tag{13}$$

$V_{eff}(x)$ is an even function, its minimum value occurs when $x = 0$, which in turn implies that $\eta = 0$. Therefore, at the end of the iterations we put $\eta = 0$.

To test the rate of convergence of AIM numerically, we calculate the angular spheroidal eigenvalue $\lambda_0^0(10)$ shown in figure 1. We simply tried a set of iteration numbers $n = 5, 10, ...$, and the convergence of AIM seems to take place smoothly when $n \geq 45$ iterations.

Proceeding in the same way, the results of the AIM for $\lambda_\ell^m(c)$ with different values of ℓ, m, and c are reported in tables 1, 2, 3, 4, and 5. The results are shown in such a way that one can easily judge the accuracy of the method. The angular spheroidal eigenvalues $\lambda_\ell^m(c)$ were calculated by means of 45 iterations only. Our calculated eigenvalues $\lambda_\ell^m(c)$ are all in excellent agreement with the available published data [8, 18, 20, 22] over the full range of parameter values, ℓ, m, and c.

A second, more stringent, test of the method is shown in Table 3, where we consider large real c, in this case, in order to reproduce more accurate results, the angular spheroidal eigenvalues $\lambda_\ell^m(c)$ were calculated by means of 100 iterations. Again the agreement is quite excellent with the available published data.

It is worthwhile to emphasize that, the AIM is very easy to implement for calculating the angular spheroidal eigenvalues $\lambda_\ell^m(c)$, without having to worry about the ranges, and forms of c. Moreover, for purely real c, and purely imaginary c, the obtained eigenvalues are satisfying a very simple ordering relation. But, for complex arbitray c values, the eigenvalues are no longer real, since the spheroidal equation is not self-adjoint. In this case the ordering is determined by the absolute magnitude of $(\lambda_\ell^m(c))^2$. Hence, one can unambiguously select the correct starting eigenvalue. This represents a significant advantage over the tridiagonal matrix method [22] in which, the eigenvalues are not ordered, and hence to choose the correct matrix eigenvalue, one must use an iterative process to move towards the starting value.

As a concluding remark, we would like to point out that, the accuracy of the results could be increased if the number of iterations are increased.

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Table 1: Comparison of selected values of eigenvalues $(\lambda_\ell^m(c))^2$ computed by Flammer [8], Le-Wei Li et al. [20], and by means of the present work.

		$(\lambda_\ell^m(c))^2$		
c^2	(m,ℓ)	Flammer [8]	Le-Wei Li et al. [20]	Present work
-1.00	(4,11)	131.560	131.560	131.560
0.10	(2,2)	6.01427	6.01427	6.01427
1.00	(1,1)	2.19555	2.19555	2.19555
	(2,2)	6.14095	6.14095	6.14095
	(2,5)	30.4361	30.4362	30.4362
4.00	(1,1)	2.73411	2.73411	2.73411
	(2,2)	6.54250	6.54250	6.54250
16.00	(1,1)	4.39959	4.39959	4.39959
	(2,5)	36.9963	36.9963	36.9963

Table 2: Comparison of selected values of eigenvalues $(\lambda_\ell^m(c))^2$ computed by Falloon et al. [22], and by means of the present work.

		$(\lambda_\ell^m(c))^2$	
c	(m,ℓ)	Falloon et al. [22]	Present work
10	(0,0)	9.228304	9.228304
	(0,1)	28.13346	28.13346
	(0,2)	45.86895	45.86895
	(1,1)	10.28777	10.28777
	(1,2)	29.33892	29.33892
	(1,3)	47.30152	47.30152
	(2,2)	13.46308	13.46308
	(2,3)	32.93818	32.93818
	(2,4)	51.52485	51.52485
$10i$	(0,0)	-81.02794	-81.02794
	(0,1)	-81.02794	-81.02794
	(0,2)	-45.48968	-45.48968
	(1,1)	-62.11935	-62.11935
	(1,2)	-62.11915	-62.11915
	(1,3)	-29.18576	-29.18576
	(2,2)	-43.29025	-43.29025
	(2,3)	-43.28716	-43.28716
	(2,4)	-13.50811	-13.50811

Table 3: Comparison of selected values of eigenvalues $(\lambda_\ell^m(c))^2$ computed by J. W. Liu [27], Falloon et al. [22], and by means of the present work.

		$(\lambda_\ell^m(c))^2$		
c	(m,ℓ)	J. W. Liu [26]	Falloon et al. [22]	Present work
50	(0,0)	49.24615	-	49.24615
	(0,1)	148.2306	-	148.2306
	(0,3)	343.1109	-	343.1109
	(0,4)	438.9725	-	438.9725
	(1,1)	50.25646	-	50.25646
	(1,2)	149.2622	-	149.2622
	(1,4)	344.1894	-	344.1894
	(1,5)	440.0769	-	440.0769
100	(0,0)	99.24810	99.24810	99.24810
	(0,1)	298.2405	298.2405	298.2405
	(0,2)	-	496.2212	496.2212
	(0,3)	693.1825	-	693.1825
	(0,4)	889.1162-	-	889.1162
	(1,1)	100.2532	100.2532	100.2532
	(1,2)	299.2558	299.2558	299.2558
	(1,3)	-	497.2472	497.2472
	(1,4)	694.2195	-	694.2195
	(1,5)	890.1645	-	890.1645

Table 4: Comparison of selected values of eigenvalues $(\lambda_\ell^m(c))^2$ computed by Le-Wei Li et al. [20], and by means of the present work.

		$(\lambda_\ell^m(c))^2$	
c	(m,ℓ)	Le-Wei Li et al. [20]	Present work
1.824770+2.601670i	(0,0)	1.701836+4.219998i	1.701836+4.219998i
2.094267+5.807965i	(0,2)	1.993901+8.576325i	1.993901+8.576325i
5.217093+3.081362i	(0,2)	23.91023+18.74194i	23.91033+18.74184i
3.563644+2.887165i	(0,1)	10.13705+11.12216i	10.13705+11.12218i
1.998555+4.097453i	(1,1)	2.919098+6.134851i	2.919095+6.134851i
3.862833+4.492300i	(1,1)	12.19691+16.24534i	12.19691+16.24534i
2.136987+5.449457i	(2,0)	6.098946+7.684379i	6.098961+7.684333i

Table 5: Comparison of selected values of eigenvalues $(\lambda_\ell^m(c))^2$ computed by B. E. Barrowes et al. [18], and by means of the present work.

c	(m,ℓ)	$(\lambda_\ell^m(c))^2$ B. E. Barrowes et al. [18]	$(\lambda_\ell^m(c))^2$ Present work
1.824770749208805+ 2.601670692890318i	(0,0)	1.705180+4.220186i	1.705180+4.220186i
3.563644553545243+ 2.887165344336900i	(0,1)	10.14084+11.12159i	10.14084+11.12159i
5.217093042404772+ 3.081362886557631i	(0,2)	23.91583+18.74332i	23.91583+18.74332i
4.067274712533398+ 6.264358978587767i	(0,3)	11.78093+22.54139i	11.78093+22.54139i
2.244329796261236+ 8.973752190228394i	(0,4)	2.156125+12.81092i	2.156124+12.81092i
7.606334073445308+ 6.906465157219409i	(0,5)	48.80665+54.01199i	48.80665+54.01197i
6.316233767329015+ 9.949229739353585i	(0,6)	29.84604+55.68999i	29.84604+55.68998i
1.998555442181652+ 4.097453662365392i	(1,0)	2.915319+6.133951i	2.915319+6.133951i
3.862833529248772+ 4.492300074953849i	(1,1)	12.20110+16.24408i	12.20110+16.24408i
2.184204069300826+ 7.326156812534641i	(1,2)	3.102506+10.53921i	3.102506+10.53921i
7.270040170458184+ 5.010809182556227i	(1,3)	47.41099+39.42618i	47.41098+39.42619i
6.119087892218941+ 8.234638882858787i	(1,4)	29.88062+46.18128i	29.88062+46.18128i
4.510843794687041+11.068777156965684i	(1,5)	14.19982+38.58584i	14.19982+38.58584i
2.136987377094029+ 5.449457313914277i	(2,0)	6.102540+7.684764i	6.102540+7.684763i
4.105156484215650+ 5.922750658440496i	(2,1)	16.13687+20.40623i	16.13687+20.40623i
5.907125751703487+ 6.283464345814330i	(2,2)	32.08759+34.48753i	32.08758+34.48754i
7.634658702877481+ 6.576768822063829i	(2,3)	53.56137+49.63852i	53.56136+49.63853i
6.337223309080594+ 9.739915760209533i	(2,4)	34.18267+53.57326i	34.18267+53.57326i
2.254441944326160+ 6.731940814252908i	(3,0)	11.27374+9.046369i	11.27374+9.046369i
4.312789375877335+ 7.267942971679416i	(3,1)	21.98994+24.06425i	21.98994+24.06425i
6.178462421804212+ 7.686866389128842i	(3,2)	38.86809+40.69989i	38.86808+40.69989i
7.954733051349589+ 8.033437876318695i	(3,3)	61.43614+58.62467i	61.43613+58.62466i
2.357663561227191+ 7.971913317957412i	(4,0)	18.43306+10.28552i	18.43306+10.28552i
4.496463440238013+ 8.560827522001995i	(4,1)	29.78282+27.39486i	29.78282+27.39486i
6.420513778117898+ 9.029346203504799i	(4,2)	47.52324+46.35817i	47.52324+46.35817i
2.450444507315414+ 9.182664291501323i	(5,0)	27.58314+11.43648i	27.58314+11.43648i
4.662342487692823+ 9.817718575239088i	(5,1)	39.52918+30.48922i	39.52918+30.48922i
2.535162563188484+10.371846133322299i	(6,0)	38.72574+12.51971i	38.72574+12.51971i

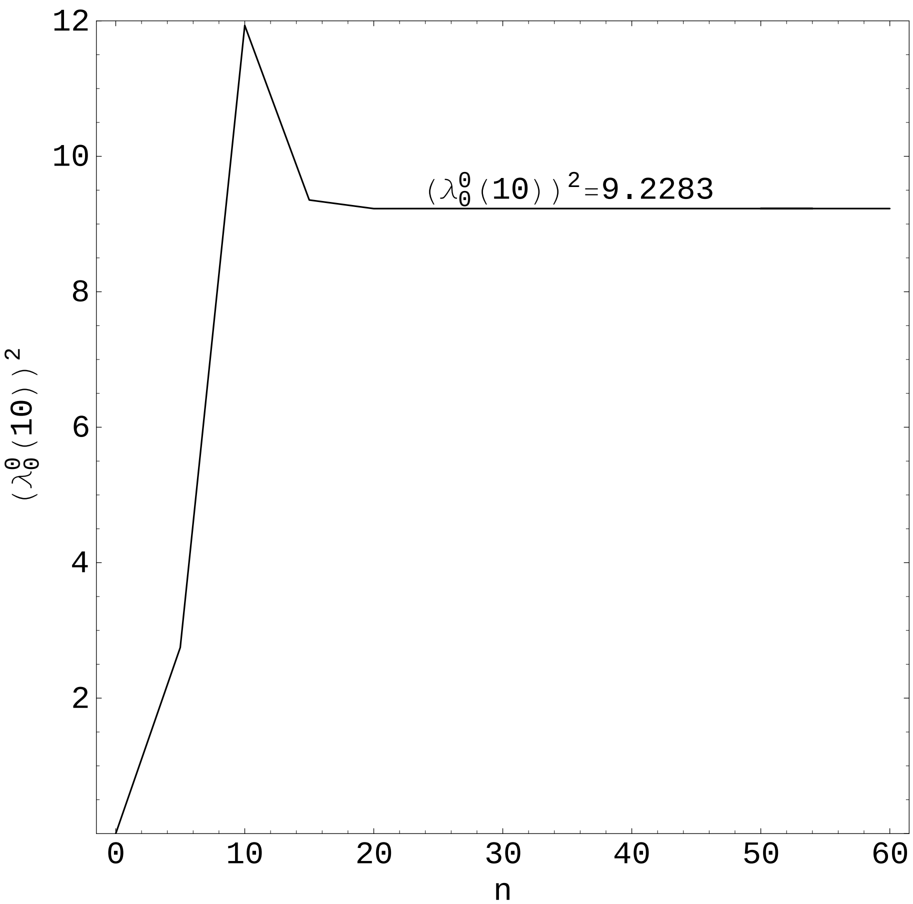

Figure 1: The rate of convergence of AIM for $\lambda_0^0(10)$ as a function of the number of iterations n.